18.06 MIDTERM 3

December 6, 2019 (50 minutes)

Please turn cell phones off completely and put them away.

No books, notes, or electronic devices are permitted during this exam.

You must show your work to receive credit. **JUSTIFY EVERYTHING**.

Please write your name on **ALL** pages that you want graded (those will be the ones we scan).

The back sides of the paper will **NOT** be graded (for scratch work only).

Do not unstaple the exam, nor reorder the sheets.

Problem 1 has $\boxed{3}$ parts, Problem 2 has $\boxed{5}$ parts, Problem 3 has $\boxed{5}$ parts.

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RECITATION INSTRUCTOR: ______________________________

SCRATCH WORK

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PROBLEM 1 **NAME:** ____________________

(1) Choose three real numbers a, b, c such that the matrices:

$$A = \begin{bmatrix} 1 & a \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & b \\ 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & c \\ 1 & 1 \end{bmatrix}$$

have the properties that:

- A has two distinct real eigenvalues
- B has two identical real eigenvalues (i.e. a repeated eigenvalue)
- C has two complex (non-real) eigenvalues

In each of these three cases, compute the eigenvalues in question. Show your work. *(15 pts)*

SCRATCH WORK

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NAME: ____________________

(2) Diagonalize A (the matrix with distinct real eigenvalues from part (1)), i.e. write it as:

$$A = VDV^{-1}$$

where V is an invertible 2×2 matrix and D is a diagonal 2×2 matrix. Explain your reasoning in figuring out V and D, and detail the step-by-step process. *(10 pts)*

SCRATCH WORK

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NAME: ____________________

(3) Recall that B has a repeated eigenvalue λ (which you should have computed in part (1)).

- Compute the eigenspace of λ, i.e. the subspace of vectors $\mathbf{v} \in \mathbb{R}^2$ such that $B\mathbf{v} = \lambda\mathbf{v}$.
- Use this to compute the geometric multiplicity of λ.
- Is B diagonalizable? *(10 pts)*

SCRATCH WORK

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PROBLEM 2 **NAME:** ____________________

Throughout this problem, the matrix A has the following Singular Value Decomposition:

$$A = \underbrace{\frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\frac{1}{3}\begin{bmatrix} 1 & x & 2 \\ 2 & 2 & y \\ 2 & -1 & -2 \end{bmatrix}}_{V^T}$$

where the matrices U and V are orthogonal and x, y denote two mystery real numbers.

(The matrices U and V include the prefactors $\frac{1}{5}$ and $\frac{1}{3}$, so the top-left entry of U is $\frac{3}{5}$ and the top-left entry of V is $\frac{1}{3}$. Recall that orthogonal means that $U^TU = I_2$ and $V^TV = I_3$)

(1) What are the values of x, y, based on the information provided? Explain how you know. *(5 pts)*

(2) Fill in the blanks (no explanation needed):

- the rank of the matrix A is ________ *(5 pts)*

- the eigenvalues of A^TA are __________, and those of AA^T are __________ *(5 pts)*

- a non-zero eigenvector of A^TA is __________ (any one eigenvector will suffice) *(5 pts)*

Hint: the answers to the blanks above are all encoded in the SVD of A

SCRATCH WORK

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NAME: ____________________

(3) Write A as a sum of two rank 1 matrices (it suffices to write these rank 1 matrices as <u>a column times a scalar times a row</u>, e.g. $\mathbf{u} \cdot \sigma \cdot \mathbf{v}^T$, you don't need to explicitly multiply the column, scalar and row out). *(5 pts)*

(4) Compute the pseudo-inverse A^+ of A, and explain how you got it (your answer for A^+ should be a 3×2 matrix with explicit numbers as entries). *(5 pts)*

SCRATCH WORK

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NAME: ____________________

(5) Use A^+ to compute a least squares solution to $A\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (i.e. you must find a vector $\mathbf{v} \in \mathbb{R}^3$ such that $A\mathbf{v}$ is as close as possible to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; explain which formula you are using).

(5 pts)

SCRATCH WORK

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PROBLEM 3 **NAME:** ____________________

(1) <u>Compute</u> the eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ of the following matrix:

$$E = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

In this problem only, you are allowed to guess the eigenvalues and eigenvectors without going through the whole process of working them out, since they are quite simple. *(10 pts)*

(2) Fill in the blank: $\mathbf{v}_1$ and $\mathbf{v}_2$ are ________________ because the matrix E is symmetric.
(5 pts)

SCRATCH WORK

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NAME: ____________________

For the remainder of this problem, consider the following setting: Alice and Bob run into Mr. Papadopoulos, who randomly chooses a letter among α, β, γ with equal probability.

- If Mr. Papadopoulos chooses α, then he gives Alice \$9 and Bob \$0
- If Mr. Papadopoulos chooses β, then he gives Alice \$0 and Bob \$9
- If Mr. Papadopoulos chooses γ, then he gives Alice \$3 and Bob \$6

Consider the random variables X_A = the amount of money Alice gets, X_B = the amount of money Bob gets, and put them in a random vector:

$$\mathbf{X} = \begin{bmatrix} X_A \\ X_B \end{bmatrix}$$

(3) Compute the expected value (a.k.a. the mean) $E[\mathbf{X}]$. *(5 pts)*

Recall that the expected value $E[X]$ of any random variable (or vector) X is the average of the possible values that X can take, weighted by the probabilities of these possible values.

SCRATCH WORK

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NAME: ____________________

(4) Compute the covariance matrix K of the random variables X_A and X_B. *(5 pts)*

Recall that the covariance of any two random variables Y and Z is the expected value:

$$E[(Y - E[Y])(Z - E[Z])]$$

The covariance matrix K of X_A and X_B is the 2×2 matrix whose entries are the covariances of the pairs of variables (X_A, X_A), (X_A, X_B), (X_B, X_A), (X_B, X_B). In terms of the vector $\mathbf{X}$ whose components are X_A and X_B, the covariance matrix K is given by the formula:

$$K = E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^T]$$

SCRATCH WORK

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NAME: ____________________

(5) Harder question: find two linear combinations of X_A and X_B (call these linear combinations Y and Z) such that the covariance of Y and Z is 0. What are the variances of Y and Z? Explain your reasoning. *(5 pts)*

SCRATCH WORK

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